The Entanglement of Superpositions

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(Dated: 5th July 2005)

Given a bipartite quantum state (in arbitrary dimension) and a decomposition of it as a superposition of two others, we find bounds on the entanglement of the superposition state in terms of the entanglement of the states being superposed. In the case that the two states being superposed are bi-orthogonal, the answer is simple, and, for example, the entanglement of the superposition cannot be more than one e-bit more than the average of the entanglement of the two states being superposed. However for more general states, the situation is very different.

PACS numbers: PACS numbers: 03.67.-a, 03.65.Ta, 03.65.Ud

INTRODUCTION

The problem we raise in this paper is the following: Given a state $|\Gamma\rangle$ of two parties, A and B, and given a certain decomposition of it as a superposition of two terms

$$|\Gamma\rangle = \alpha|\Psi\rangle + \beta|\Phi\rangle \ , \tag{1}$$

what is the relation between the entanglement of $|\Gamma\rangle$ and those of the two terms in the superposition? Given how central entanglement is to quantum mechanics, and how central superposition is to entanglement, this question seems to be a basic one; as far as we are aware, however, little is known about it. This is particularly surprising for bipartite pure states, as for them at least the measure of entanglement is completely understood–the entanglement of a bipartite pure state is the von Neumann entropy of the reduced state of either of the parties [1]:

$$E(\Psi) \equiv S\big(\mathrm{Tr}_A|\Psi\rangle\langle\Psi|\big) = S\big(\mathrm{Tr}_B|\Psi\rangle\langle\Psi|\big) \tag{2}$$

Before embarking on our study, it is worth making some observations. To start with, at first sight it seems unlikely that there could be any relation at all. Indeed, entanglement is a global property of a state, originating precisely from the superposition of different terms; looking at each term separately seems completely to miss the point. For example, consider a state of two qubits

$$|\gamma\rangle = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle. \tag{3}$$

Each term by itself is unentangled, yet their superposition is a maximally entangled state of the qubits. On the other hand, consider

$$|\gamma'\rangle = \frac{1}{\sqrt{2}}|\Phi^+\rangle + \frac{1}{\sqrt{2}}|\Phi^-\rangle \tag{4}$$

where

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}|0\rangle|0\rangle \pm \frac{1}{\sqrt{2}}|1\rangle|1\rangle \tag{5}$$

Each of the terms in the superposition is maximally entangled, yet the superposition itself is unentangled.

We also note that in general, two states of high fidelity to one another–i.e. *they are almost the same state*–do not necessarily have nearly the same entanglement. That is when $|\langle\psi|\phi\rangle|^2 \to 1$ in general it is not true that $E(\psi) \to E(\phi)$.

For example let

$$\begin{aligned} |\phi\rangle &= |0\rangle|0\rangle \ \text{and} \\ |\psi\rangle &= \sqrt{1-\epsilon}\,|\phi\rangle + \sqrt{\frac{\epsilon}{d}}\Big[|1\rangle|1\rangle + |2\rangle|2\rangle \ldots + |d\rangle|d\rangle\Big] \ . \end{aligned} \tag{6}$$

In this case $E(\phi) = 0$ but

$$E(\psi) = -(1-\epsilon)\log_2(1-\epsilon) - d\left(\frac{\epsilon}{d}\log_2\frac{\epsilon}{d}\right) \approx \epsilon \log_2 d \ . \tag{7}$$

The fidelity $|\langle\psi|\phi\rangle|^2 = 1-\epsilon$ approaches one for small ϵ, but for any ϵ we can pick a d such that the difference in the entanglements of $|\phi\rangle$ and $|\psi\rangle$ is however large we like. The amount the entanglement of two states of fixed dimension can differ as a function of fidelity is bounded using Fannes's inequality [3]. In infinite dimensions no such bound applies and entanglement is not a continuous function.

On the other hand, suppose that we have a state with large number of Schmidt terms in its decomposition. It is obvious that by adding a small number of supplementary terms with small overall weight (and then normalize the resulting state) one cannot affect the overall entanglement too much. This leads us to think that despite

the previous arguments, there is a relation between the entanglement of a state and the individual terms that by superposition yield the state.

BI-ORTHOGONAL STATES

The simplest case is when the two states we are superposing, $|\Phi_1\rangle$ and $|\Psi_1\rangle$ are bi-orthogonal, i.e.

$$\begin{aligned}\mathrm{Tr}_A\Big(\mathrm{Tr}_B(|\Phi_1\rangle\langle\Phi_1|)\mathrm{Tr}_B(|\Psi_1\rangle\langle\Psi_1|)\Big) &= \\ \mathrm{Tr}_B\Big(\mathrm{Tr}_A(|\Phi_1\rangle\langle\Phi_1|)\mathrm{Tr}_A(|\Psi_1\rangle\langle\Psi_1|)\Big) &= 0. \qquad (8)\end{aligned}$$

Up to local unitary transformations,

$$\begin{aligned}|\Phi_1\rangle &= \sum_{i=1}^{d_1} a_i\,|i\rangle|i\rangle \\ |\Psi_1\rangle &= \sum_{i=d_1+1}^{d} b_i\,|i\rangle|i\rangle\,, \qquad (9)\end{aligned}$$

where a_i and b_i are positive and real. Since Alice's reduced states for $|\Phi_1\rangle$ and $|\Psi_1\rangle$ are diagonal in the same basis, is not difficult to calculate directly that the entanglement of the superposition $|\Gamma_1\rangle = \alpha\,|\Phi_1\rangle + \beta\,|\Psi_1\rangle$ is given by

$$E(\Gamma_1) = |\alpha|^2E(\Phi_1) + |\beta|^2E(\Psi_1) + h_2(|\alpha|^2), \qquad (10)$$

where $h_2(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function, and we take $|\alpha|^2 + |\beta|^2 = 1$.

In fact the following inequalities hold for any density matrices [2]; these will be used repeatedly in what follows ($S(\rho)$ denotes the von Neumann entropy of ρ):

$$|\alpha|^2S(\rho) + |\beta|^2S(\sigma) \le S(|\alpha|^2\rho + |\beta|^2\sigma) \qquad (11)$$

and

$$S(|\alpha|^2\rho + |\beta|^2\sigma) \le |\alpha|^2S(\rho) + |\beta|^2S(\sigma) + h_2(|\alpha|^2). \quad (12)$$

There is equality in (12) if and only if ρ and σ are orthogonal. Since $|\Phi_1\rangle$ and $|\Psi_1\rangle$ are bi-orthogonal, their reduced density matrices are orthogonal so we could have used (12) rather than direct calculation to give (10).

Let us use the following notation for the expression on the right-hand-side of equation (10):

$$\Upsilon(\Phi,\Psi,\alpha) \equiv |\alpha|^2E(\Phi) + |\beta|^2E(\Psi) + h_2(|\alpha|^2). \quad (13)$$

Thus for bi-orthogonal states, the ratio

$$\frac{E(\Gamma_1)}{\Upsilon(\Phi_1,\Psi_1,\alpha)} = 1\ . \qquad (14)$$

In addition the maximum increase of entanglement is bounded:

$$E(\Gamma_1) - \Big(|\alpha|^2E(\Phi_1) + |\beta|^2E(\Psi_1)\Big) \le 1, \qquad (15)$$

independent of the dimension.

We also point out that if we *mix* rather than superpose two pure states, the entanglement of formation [4, 5] is at most the average of the entanglement of the individual states.

However we will soon see that any intuition we might have gained by considering the case of bi-orthogonal states is misleading.

ORTHOGONAL (BUT NOT NECESSARILY BI-ORTHOGONAL) STATES

We now prove the following result. Given two states $|\Phi_2\rangle$ and $|\Psi_2\rangle$ which are orthogonal but not necessarily bi-orthogonal, the entanglement of the superposition

$$|\Gamma_2\rangle = \alpha\,|\Phi_2\rangle + \beta\,|\Psi_2\rangle\,, \qquad (16)$$

(where $|\alpha|^2+|\beta|^2 = 1$, so that $|\Gamma_2\rangle$ is normalized) satisfies

$$\begin{aligned}&E(\alpha\Phi_2 + \beta\Psi_2) \\ &\quad\le 2\Big(|\alpha|^2E(\Phi_1) + |\beta|^2E(\Psi_1) + h_2(|\alpha|^2)\Big). \quad (17)\end{aligned}$$

To prove this, consider that Alice, in addition to Hilbert space $\mathcal{H}_A$, has a qubit with Hilbert space denoted $\mathcal{H}_a$. And consider the state

$$|\Delta_2\rangle = \alpha|0\rangle_a\,|\Phi_2\rangle_{AB} + \beta|1\rangle_a\,|\Psi_2\rangle_{AB}\,. \qquad (18)$$

Bob's reduced state for $|\Delta_2\rangle$ is

$$\rho_B = |\alpha|^2\mathrm{Tr}_A\Big(\,|\Phi_2\rangle\langle\Phi_2|\,\Big) + |\beta|^2\mathrm{Tr}_A\Big(\,|\Psi_2\rangle\langle\Psi_2|\,\Big). \quad (19)$$

The inequality (12) shows that

$$\begin{aligned}S(\rho_B) \le\ & |\alpha|^2S\big(\mathrm{Tr}_A|\Phi_2\rangle\langle\Phi_2|\,\big) \qquad (20)\\ &+ |\beta|^2S\big(\mathrm{Tr}_A|\Psi_2\rangle\langle\Psi_2|\,\big) + h_2(|\alpha|^2).\end{aligned}$$

However ρ_B may also be written

$$\begin{aligned}\rho_B =\ & \frac{1}{2}\mathrm{Tr}_A\Big[\Big(\alpha\,|\Phi_2\rangle + \beta\,|\Psi_2\rangle\Big)\Big(\bar\alpha\,\langle\Phi_2| + \bar\beta\,\langle\Psi_2|\Big)\Big] \quad (21)\\ &+\frac{1}{2}\mathrm{Tr}_A\Big[\Big(\alpha\,|\Phi_2\rangle - \beta\,|\Psi_2\rangle\Big)\Big(\bar\alpha\,\langle\Phi_2| - \bar\beta\,\langle\Psi_2|\Big)\Big].\end{aligned}$$

Thus (11), shows that

$$\begin{aligned}&\frac{1}{2}S\Big(\mathrm{Tr}_A\Big[\Big(\alpha\,|\Phi_2\rangle + \beta\,|\Psi_2\rangle\Big)\Big(\bar\alpha\,\langle\Phi_2| + \bar\beta\,\langle\Psi_2|\Big)\Big]\Big) \\ &\quad+\frac{1}{2}S\Big(\mathrm{Tr}_A\Big[\Big(\alpha\,|\Phi_2\rangle - \beta\,|\Psi_2\rangle\Big)\Big(\bar\alpha\,\langle\Phi_2| - \bar\beta\,\langle\Psi_2|\Big)\Big]\Big) \\ &\quad\le S(\rho_B). \qquad (22)\end{aligned}$$

Thus

$$\begin{aligned}&\frac{1}{2}E(\alpha\Phi_2 + \beta\Psi_2) + \frac{1}{2}E(\alpha\Phi_2 - \beta\Psi_2) \\ &\quad\le \Big(|\alpha|^2E(\Phi_1) + |\beta|^2E(\Psi_1) + h_2(|\alpha|^2)\Big). \quad (23)\end{aligned}$$

Since $E(\alpha\Phi_2 - \beta\Psi_2) \geq 0$, we deduce the advertised inequality (17).

The inequality may also be written

$$\frac{E(\alpha\Phi_2 + \beta\Psi_2)}{\Upsilon(\Phi_2, \Psi_2, \alpha)} \leq 2. \tag{24}$$

One may wonder whether the factor of two on the right-hand-side of this equation is an artifact of our proof, and whether in fact the factor should be one as in (14). As we now show, even for qubits, one can get as close as we wish to the ratio two in this equation. For consider the following choices:

$$\begin{aligned} |\phi_2\rangle &= |0\rangle|0\rangle \\ |\psi_2\rangle &= \sqrt{y/2}\,|0\rangle|1\rangle + \sqrt{y/2}\,|1\rangle|0\rangle - \sqrt{1-y}\,|1\rangle|1\rangle \\ \alpha &= xy; \quad \beta = \sqrt{1-\alpha^2} \end{aligned} \tag{25}$$

where x and y are real parameters. We are interested in the behavior of this family of states as y tends to zero, with x fixed. $|\phi_2\rangle$ is unentangled, and as y tends to zero (with x fixed), $|\psi_2\rangle$ and $|\gamma_2\rangle = \alpha\,|\phi_2\rangle + \beta\,|\psi_2\rangle$ both get closer and closer to being unentangled. It is not difficult to check that

$$\lim_{y\to 0} \frac{E(\alpha\phi_2 + \beta\psi_2)}{\Upsilon(\phi_2, \psi_2, \alpha)} = \frac{(1+2x)^2}{1+4x^2}. \tag{26}$$

We note that this limit is 2 for $x = 1/2$.

Since the states in this case are close to being unentangled, the example might seem to be a trick of the limiting behavior and possibly uninteresting. It might be thought that one can only achieve equality in the bound (24) for essentially unentangled states, and that the *increase* in entanglement could never violate the bound (15). However, in larger dimensions than qubits this is not the case. Consider the following example when Alice and Bob both have Hilbert spaces of dimension d:

$$\begin{aligned} |\phi_2'\rangle &= \frac{1}{\sqrt{2}}\Big(|1\rangle|1\rangle + \frac{1}{\sqrt{d-1}}\Big[|2\rangle|2\rangle + |3\rangle|3\rangle \ldots |d\rangle|d\rangle\Big]\Big) \\ |\psi_2'\rangle &= \frac{1}{\sqrt{2}}\Big(|1\rangle|1\rangle - \frac{1}{\sqrt{d-1}}\Big[|2\rangle|2\rangle + |3\rangle|3\rangle \ldots |d\rangle|d\rangle\Big]\Big) \\ \alpha &= -\beta = \frac{1}{\sqrt{2}} \end{aligned} \tag{27}$$

The entanglement of $|\phi_2'\rangle$ and $|\psi_2'\rangle$ is $\frac{1}{2}\log_2(d-1)+1$; and the entanglement of $\alpha\,|\phi_2'\rangle + \beta\,|\psi_2'\rangle$ is $\log_2(d-1)$. Thus

$$\frac{E(\alpha\phi_2' + \beta\psi_2')}{\Upsilon(\phi_2', \psi_2', \alpha)} \to 2, \tag{28}$$

as $d \to \infty$ and the increase in entanglement is

$$\begin{aligned} E(\alpha\phi_2' + \beta\psi_2') \; - \; \Big(|\alpha|^2 E(\phi_2') + |\beta|^2 E(\psi_2')\Big) \\ = \frac{1}{2}\log_2(d-1) - 1. \end{aligned} \tag{29}$$

Thus the increase in entanglement can be greater than one e-bit, and in fact unbounded. For this example the increase in entanglement is only greater than one e-bit for $d > 17$. However, using numerical searches, we have found examples even for $d = 3$ for which the increase in entanglement is more than one e-bit.

ARBITRARY STATES

The most general case, when the two states we are superposing are non-orthogonal, is also interesting. In this case we may prove the following inequality: Let $|\Phi_3\rangle$ and $|\Psi_3\rangle$ be normalized but otherwise arbitrary, and as before we take $|\alpha|^2 + |\beta|^2 = 1$. Then

$$\begin{aligned} \|\alpha\,|\Phi_3\rangle - \beta\,|\Psi_3\rangle\,\|^2 E(\alpha\Phi_3 + \beta\Psi_3) \leq \\ 2\Big(|\alpha|^2 E(\Phi_3) + |\beta|^2 E(\Psi_3) + h_2(|\alpha|^2)\Big) \end{aligned} \tag{30}$$

The notation $E(\alpha\Phi_3 + \beta\Psi_3)$ denotes the entanglement of the normalized version of the state $\alpha\,|\Phi_3\rangle + \beta\,|\Psi_3\rangle$.

To prove (30) again let us consider an expression of the form (18)

$$|\Delta_3\rangle = \alpha|0\rangle_a\,|\Phi_3\rangle_{AB} + \beta|1\rangle_a\,|\Psi_3\rangle_{AB}\,. \tag{31}$$

Although $|\Delta_3\rangle$ is normalized, the state

$$|\Gamma_3\rangle = \alpha\,|\Phi_3\rangle + \beta\,|\Psi_3\rangle \tag{32}$$

need not be.

Now, as before, Bob's reduced state for $|\Delta_3\rangle$ can be written in two ways:

$$\rho_B = |\alpha|^2 \mathrm{Tr}_A\Big(|\Phi_3\rangle\langle\Phi_3|\Big) + |\beta|^2 \mathrm{Tr}_A\Big(|\Psi_3\rangle\langle\Psi_3|\Big) \tag{33}$$

and

$$\begin{aligned} \rho_B \;=\; & \frac{\|\alpha\,|\Phi_3\rangle + \beta\,|\Psi_3\rangle\,\|^2}{2} \mathrm{Tr}_A\Big[\Big(\frac{\alpha\,|\Phi_3\rangle + \beta\,|\Psi_3\rangle}{\|\alpha\,|\Phi_3\rangle + \beta\,|\Psi_3\rangle\,\|}\Big)\Big(\frac{\bar\alpha\,\langle\Phi_3| + \bar\beta\,\langle\Psi_3|}{\|\alpha\,|\Phi_3\rangle + \beta\,|\Psi_3\rangle\,\|}\Big)\Big] \\ & + \frac{\|\alpha\,|\Phi_3\rangle - \beta\,|\Psi_3\rangle\,\|^2}{2} \mathrm{Tr}_A\Big[\Big(\frac{\alpha\,|\Phi_3\rangle - \beta\,|\Psi_3\rangle}{\|\alpha\,|\Phi_3\rangle - \beta\,|\Psi_3\rangle\,\|}\Big)\Big(\frac{\bar\alpha\,\langle\Phi_3| - \bar\beta\,\langle\Psi_3|}{\|\alpha\,|\Phi_3\rangle - \beta\,|\Psi_3\rangle\,\|}\Big)\Big]. \end{aligned}$$

We have explicitly written ρ_B as a mixture of trace one operators. So now using (11) and (12) we deduce that

$$\|\alpha\,|\Phi_3\rangle + \beta\,|\Psi_3\rangle\,\|^2 E(\alpha\Phi_3 + \beta\Psi_3) \leq 2\Big(|\alpha|^2 E(\Phi_3) + |\beta|^2 E(\Psi_3) + h_2(|\alpha|^2)\Big), \tag{34}$$

or equivalently

$$\frac{E(\alpha\Phi_3 + \beta\Psi_3)}{\Upsilon(\Phi_3, \Psi_3, \alpha)} \leq \frac{2}{\|\alpha\,|\Phi_3\rangle + \beta\,|\Psi_3\rangle\,\|^2}. \tag{35}$$

We do not know whether this bound is the best possible; we suspect not. However, unlike the case where the two superposed states are orthogonal, for which the ratio

$$\frac{E(\alpha\Phi_3 + \beta\Psi_3)}{\Upsilon(\Phi_3, \Psi_3, \alpha)} \tag{36}$$

is bounded by two, this ratio is unbounded for non-orthogonal states. For consider

$$\begin{aligned} |\phi_3\rangle &= |1\rangle|1\rangle \\ |\psi_3\rangle &= \sqrt{1-\epsilon}\,|1\rangle|1\rangle - \frac{\epsilon}{\sqrt{d}}\Big[\,|1\rangle|1\rangle + |2\rangle|2\rangle \ldots |d\rangle|d\rangle\,\Big] \\ \alpha &= \frac{\sqrt{1-\epsilon}}{\sqrt{2-\epsilon}}; \quad \beta = \frac{-1}{\sqrt{2-\epsilon}}. \end{aligned} \tag{37}$$

In this case

$$\frac{\alpha\,|\phi_3\rangle + \beta\,|\psi_3\rangle}{\|\alpha\,|\phi_3\rangle + \beta\,|\psi_3\rangle\,\|} = \frac{1}{\sqrt{d}}\Big[\,|1\rangle|1\rangle + |2\rangle|2\rangle \ldots |d\rangle|d\rangle\,\Big] \tag{38}$$

So $E(\alpha\phi_3 + \beta\psi_3) = \log_2 d$, and for fixed d, we can let ϵ be as small as we like so that $E(\psi_3) \approx 0$ and $h_2(\alpha^2) \approx 1$. Hence as $\epsilon \to 0$,

$$\frac{E(\alpha\phi_3 + \beta\psi_3)}{\Upsilon(\phi_3, \psi_3, \alpha)} \to \log_2 d. \tag{39}$$

This example is also interesting since the increase in entanglement

$$E(\alpha\phi_3 + \beta\psi_3) - \Big(|\alpha|^2 E(\phi_3) + |\beta|^2 E(\psi_3)\Big) \to \log_2 d, \tag{40}$$

as $\epsilon \to 0$, which is the maximum possible increase in dimension d. Notice that the trick used here is quite similar to that used in (6) which exhibits two states of high fidelity but very different entanglement. Here, we take ϵ still smaller, resulting in two states of high fidelity, nearly the same entanglement, but vastly different Schmidt ranks. The large increase of entanglement comes about when the normalization of the superposition $|\gamma_3\rangle = \alpha\,|\phi_3\rangle + \beta\,|\psi_3\rangle$ supplies weight to those many Schmidt terms.

We end by noting that the methods we have used yield straightforward generalizations of our results to cases where there are more than two terms in the superposition.

NL and SP thank the EU for support through the European Commission project RESQ (contract IST-2001-37559); NL, SP and JAS thank the UK EPSRC for support through the Interdisciplinary Research Collaboration in Quantum Information Processing; JAS thanks the US National Security Agency and the Advanced Research and Development Activity for support through contract DAAD19-01-C-0056.

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